DIFFERENTIAL EQUATION MODELS IN BIO-MEDICAL TECHNOLOGY

A general system of first order ordinary differential equations with the generic equation in the form

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \ldots, x_n) \qquad (1)$$

can be compactly represented in the vector form

$$\frac{d\bar{x}}{dt} = \bar{f}(t, \bar{x}) \qquad (2)$$

A case of importance in applications occurs when f does not involve the (time) variable t explicitly and then the system is called autonomous. Thus

$$\frac{d\bar{x}}{dt} = \bar{f}(\bar{x}) \qquad (3)$$

is an autonomous system and $\bar{f}$ is a real function in an n-dimensional domain D in x-space. An autonomous system is invariant under translations of the independent variable (t in (3)). Solutions of (3) are represented by curves in n-space. Let $\bar{\phi}(t, \bar{\eta})$ be the solution of (3) satisfying the initial condition $\bar{\phi}(t_o, \bar{\eta}) = \bar{\eta}$ = a point in the domain D. Then $\bar{\phi}$ represents the motion of a particle in D moving with velocity $\bar{f}$ and passing through $\bar{\eta}$ in D at time t_o. This motion is completely characterized by a curve C in the region D and C is given by the parametric equation $\bar{x} = \bar{\phi}(t, \bar{\eta})$ with t as a parameter. The equation (3) represents the motion of a dynamical system of n degrees of freedom and also indicates the relationship between the velocity vector and the displacement vector of a generic particle moving in D. The displacement - velocity

plane/hyperplane is the Poincaré phase plane. If t_o is a constant, $\bar{\phi}(t + t_o, \bar{\eta})$ is also a solution of (3) and the two solutions $\bar{\phi}(t, \bar{\eta})$, $\bar{\phi}(t + t_o, \bar{\eta})$ represent two motions with a phase difference t_o. The orbit or solution curve C in the region D corresponding to the two motions is the same. For a given orbit C in the domain D there is a one-parameter family of solutions and any two motions of the family differ by a phase. A point $\bar{a}$ in the domain D at which $\bar{f}$ vanishes (i.e. $\bar{f}(\bar{a}) = \bar{0}$) is a critical point of the system (3). The constant vector $\bar{x} = \bar{\phi}(t) = \bar{a}$ is a solution of (3) and the orbit of this solution consists of a single point a of D in phase space.

D'Ancona-Volterra model of the shark-food fish community

$$\frac{dx}{dt} = ax - bxy, \quad \frac{dy}{dt} = -cy + dxy \qquad (4)$$

The system (4) governs the interaction of the selachians and food fish in the absence of fishing.

Prey population x(t) grows at the constant rate a (> 0) in the absence of the selachians. The number of contacts per unit time between predators and prey is bxy (b > 0). Predator population y(t) has a natural rate of decrease c in the absence of the prey fish and increases at a rate dxy proportional to the present amount y and the food supply x. Thus the model (4) above in which (a, b, c, d) are positive constants.

The system (4) has the obvious solutions (x(t) = 0, y(t) = 0) and (x(t) = c/d, y(t) = a/b). The trivial solution at the origin (x = 0, y = 0) is of no particular interest. The solutions (x(t)

$= x_o\, e^{at}$, $y(t) = 0$) and ($x(t) = 0$, $y(t) = y_o\, e^{-ct}$) show that the two axes of x and y are orbits of (4) and hence every solution of (4) that starts in the first quadrant $x > 0$, $y > 0$ at time $t = t_o$ will remain in the quadrant for all future time $t \geq t_o$.

The orbits of (4) for ($x, y \neq 0, 0$) are the solutions of the first order equation

$$\frac{dx}{dx} = \frac{-cy + dxy}{ax - bxy} = \frac{y\,(-c + dx)}{x\,(a - by)} \qquad (5)$$

This equation is separable in the variables x, y and on integration we have

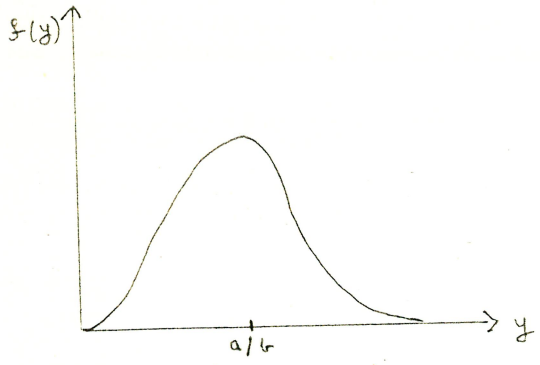
$$\left(\frac{y^a}{e^{by}}\right)\left(\frac{x^c}{e^{dx}}\right) = A \qquad (6)$$

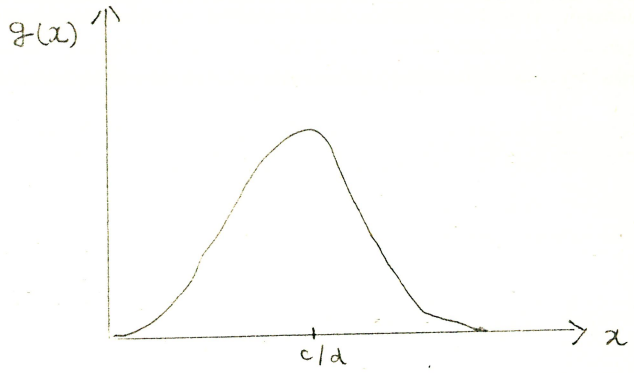
A being a constant of integration. The function $f(y) = (y^a/e^{by})$ has the property that $f(o) = 0$, $f(\infty) = 0$ and $f(y) > 0$ for $y > 0$. Also f(y) increases from $y = 0$ to $y = a/b$ and then decreases. $f(y) = (a/b)^a/e^a$ at $y = a/b$. The function $g(x) = (x^c)/(e^{dx})$ has the property that $g(o) = 0$, $g(\infty) = 0$ and g(x) increases from $x = 0$ to $x = c/d$ and then decreases

$$M_x = \max g(x) = g\left(\frac{c}{d}\right) = \left(\frac{c}{d}\right)^c e^{-c}$$

$$M_y = \max f(y) = f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^a e^{-a}$$

The equation (6) has no solution in the first quadrant of the phase plane xy for $A > M_x\, M_y$ and there is the single solution $x = c/d$, $y = a/b$ for $A = M_x\, M_y$. Let $A = \lambda M_y$ with $0 < \lambda < M_x$. The equation $g(x) = \lambda$ has one solution $x = x_m < c/d$ and one solution $x = x_M > c/d$.





$f(y) = y^a e^{-by}$ $(a, b > 0)$ $\qquad$ $g(x) = x^c e^{-dx}$ $(c, d > 0)$

(Figure 1)

The equation $f(y) = \lambda M_y / g(x)$ has no solution when x is less than x_m and when x is greater than x_M. When $x = x_m$ or x_M there is just one solution $y = a/b$. When x lies between x_m and x_M there are two solutions $y_1(x)$ and $y_2(x)$. Let $y_1(x) < y_2(x)$. It is seen that $y_1(x) < a/b < y_2(x)$ and when $x \to x_m$ or $x \to x_M$, both $y_1(x)$ and $y_2(x) \to a/b$. The curves (6) viz.,

$$f(y)\, g(x) = A \qquad (6)$$

are closed for $x > 0$, $y > 0$ (Figure 2).

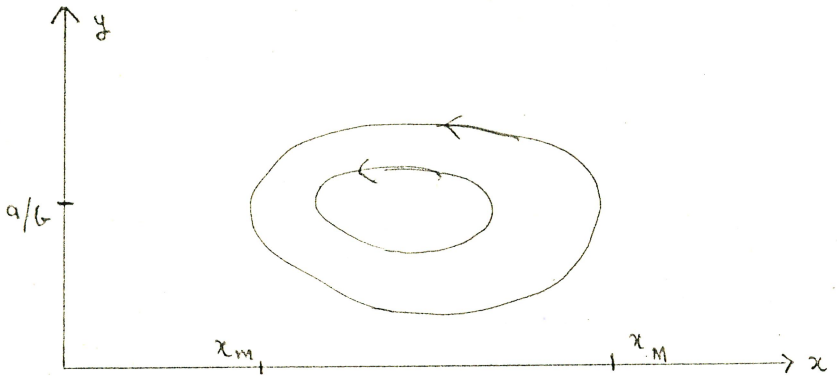


Figure 2. Orbits of (4) for $x > 0$, $y > 0$

The solution $x = c/d$, $y = a/b$ is a point loop and is an equilibrium position of (4). Barring this there is no equilibrium position on any of the closed curves (6). The solutions $x(t)$, $y(t)$ of (4) with $x(o) > 0$, $y(o) > 0$ are periodic functions of time. $x(t + T) = x(t)$, $y(t + T) = y(t)$ with T positive if $x(o) > 0$, $y(o) > 0$.

Let $\bar{x}$, $\bar{y}$ be the averages of $x(t)$, $y(t)$ over the period.

$$T\bar{x} = \int_0^T x(t)\,dt, \quad T\bar{y} = \int_0^T y(t)\,dt \tag{7}$$

$$\int_0^T \frac{\dot{x}(t)}{x(t)}\,dt = \log x(T) - \log x(0) = 0 \text{ since}$$

$x(T) = x(0)$. Since $\dot{x}(t) = ax - bxy$ (4),

$$\int_0^T \frac{\dot{x}(t)}{x(t)}\,dt = \int_0^T (a - by)\,dt = aT - bT\bar{y}$$

and hence $\bar{y} = a/b$. Similarly $\bar{x} = c/d$.
The average values of $x(t)$, $y(t)$ over the period are the equilibrium values.

The system (4) governs the interaction of the predators and food fish in the absence of fishing. To include the effects of fishing in the model we may note that fishing decreases the population of food fish at a rate $\varepsilon\, x(t)$ and the population of selachians at a rate $\varepsilon\, y(t)$. The constant ε (> 0) is a measure of the intensity of fishing.

$$\frac{dx}{dt} = (a - \varepsilon)\,x - bxy, \quad \frac{dy}{dt} = -(c + \varepsilon)\,y + dxy \tag{8}$$

This system (8) is the same as (4) it $a - \varepsilon > 0$ with a and c replaced by $a - \varepsilon$ and $c + \varepsilon$ respectively. Hence

$$\bar{x} = \frac{c+\varepsilon}{d}, \quad \bar{y} = \frac{a-\varepsilon}{b} \tag{9}$$

Therefore a <u>moderate</u> amount of fishing ($\varepsilon < a$) increases the (average) number of fish and decreases the (average) number of selachians. A reduced level of fishing (a reduction in ε)

increases the (average) number of selachians and decrease the (average) number of food fish.

The following data gives the percentages of total catch of selachians (sharks, skates, rays, etc. - predators) for the port of Fiume, Italy, studied by the Italian biologist umberto D' Ancona.

Year	1914	1915	1916	1917	1918	1919	1920
0/0	11.9	21.4	22.1	21.2	36.4	27.3	16.0

1921	1922	1923
15.9	14.8	10.7

The large increase in the percentage of selachians during the war years was indeed a puzzle and the models (4), (8) proposed by the mathematician Volterra (1931) can explain the reason therefor. Hence the following principle of Volterra:

A moderate amount of fishing increases the average number of food fish and decrease the average number of selachians. A reduced level of fishing increases the average number of selachians and decreases the average number of food fish.

It may well be believed that the Great War (1914 - 1918) compelled a reduction in the level of fishing in the Mediterranean sea.

Volterra's principle has spectacular applications to insecticide treatments that destroy both insect predators and their insect prey. The application of insecticides will actually increase the population of those insects that are kept in control by other predatory insects ($\bar{x}_2 = (a+\varepsilon)/(d) > \bar{x}_1 = a/d$). After the

discovery of DDT, spraying of this pesticide to kill the scale insects in orchards resulted in the unwelcome realization of an increase of the scale insects, conforming to Volterra's principle. However, the models (4), (8) are not favoured by many ecologists and biologists.

A more general model of predator - prey interactions is the system of differential equations

$$\frac{dx}{dt} = ax - bxy - ex^2, \quad \frac{dy}{dt} = -cx + dxy - fy^2 \qquad (10)$$

The terms ex^2, fy^2 reflect internal competition of the prey (x) and the predator (y) respectively. In general, the system (10) does not possess periodic solutions and all solutions approach equilibrium solutions if both e and f are positive. Even this model (10) is not readily accepted by ecologists and biologists. No solution of (10) with $x(o)\ y(o) \neq 0$ can reach $x = 0$ or $y = 0$ in finite time.

Quadratic Population Models

Let x(t), y(t) denote the population levels (i.e., the number of individuals) of two interacting species at time t. Though the values of x(t), y(t) are integers and change by integer amounts, for large populations change by one or two over a short time span is `infinitesimal' relative to the total and a continuous change of populations is thus conceivable. In the quadratic model

$$\frac{dx}{dt} = (a_o + a_1 x + a_2 y)x \ , \qquad (11)$$

$$\frac{dy}{dt} = (b_o + b_1 x + b_2 y)y \qquad (11)$$

let a_o, a_1, a_2, b_o, b_1, b_2 be constant.

If $a_1 = a_2 = b_1 = b_2 = 0$, we have a non-interacting pair of species in which both $x(t) = x(o)\, e^{a_o t}$, $y(t) = y(o)\, e^{b_o t}$ change exponentially with time. If $a_2 = 0$, $b_1 = 0$, we have again a pair of noninteracting species, each of which follows a generalized logistic law, viz.,

$$\frac{dx}{dt} = (a_o + a_1 x)x, \quad \frac{dy}{dt} = (b_o + b_1 y)\, y.$$

If $a_1 = b_2 = 0$, (11) reduces to D'Ancona - Volterra model (4) of shark-food fish interactions.

The constants a_o, b_o in (11) represent the natural growth of the respective species in the absence of other interactions and the terms $a_o x$, $b_o y$ are called the <u>biotic potentials</u> of the species. The constant a_1 measures the effect of the population $x(t)$ on the growth rate of species x. If $a_1 < 0$, a_1 is the self-limiting coefficient measuring the negative effects of overpopulation. If $a_1 > 0$, it is the mutualism coefficient and indicates an increase in productivity with an increase in population. It is a case of competition among the members of the species x for limited resources if $a_1 < 0$ and of cooperation if $a_1 > 0$. Similar observations hold for b_2. The coefficients a_2 and b_1 measure the effect of each species on the growth rate of the other and are called two-species social coefficients. Let $a_2 b_1 \neq 0$.

(i) $a_2 > 0$, $b_1 > 0$ is the case of interspecies mutualism.

(ii) $a_2 < 0$, $b_1 > 0$ or $a_2 > 0$, $b_1 < 0$. Then x is the prey and y the predator or vice-versa.

(iii) $a_2 < 0$, $b_1 < 0$ is the case of mutual predation or competition for common resources.

Analytical determination of x(t), y(t) from (11) is not possible in the absence of information about the coefficients. Two interacting species surviving over a long time in common environment may well be in a state of balance or equilibrium. In such a case the growth rates dx/dt, dy/dt are both zero and the critical points of (11) are given by

$$x(a_o + a_1x + a_2y) = 0,\ y(b_o + b_1x + b_2y) = 0$$

The three critical points or constant population values are

$$(x = 0,\ y = 0),\ (x = 0,\ y = -b_o/b_2),$$

$$(x = -a_o/a_1,\ y = 0)$$

and in each of these at least one of the species is extinct. Thus none of these deserves attention in the wake of long term balance between the two species. Another critical point exists if $a_1b_2 \neq a_2b_1$ and then

$$x_o = \frac{a_2\, b_2 - a_o\, b_2}{a_1\, b_2 - a_2\, b_1},\quad y_o = \frac{a_o\, b_1 - a_1\, b_o}{a_1\, b_2 - a_2\, b_1}$$

This represents a true-population state only if $x_o > 0$, $y_o > 0$.

Another kind of simple equilibrium state exists with x(t), y(t) being nonconstant periodic functions of time with a common period. The population curve / orbit / trajectory is then a cycle. Let us impose the two following conditions on the coefficients of (11).

$A = a_1b_2 - a_2b_1 \neq 0$; $B = a_1b_o\,(a_2 - b_2) - a_o\,b_2\,(a_1 - b_1) \neq 0$. The condition on A means the straightlines $a_o + a_1x + a_2y =$ constant, $b_o + b_1x + b_2y =$ constant are not parallel. The condition on B involves all the coefficients of (11).

<u>No cycles theorem</u>. Under the conditions $A \neq 0$, $B \neq 0$, the quadratic population model (11) has no cycles in the (first) quadrant.

Proof : For $x > 0$, $y > 0$ let $k(x, y) = x^{\alpha}\, y^{\beta}$

with $\alpha = \dfrac{b_2\,(b_1 - a_1)}{A} - 1$, $\beta = \dfrac{a_1\,(a_2 - b_2)}{A} - 1$

Denoting the system (11) as $\dfrac{dx}{dt} = f$, $\dfrac{dy}{dt} = g$

we have

$$f = x(a_o + a_1x + a_2y),\ g = y(b_o + b_1x + b_2y)$$

and it can be seen easily that

$$\frac{\partial}{\partial x}(kf) + \frac{\partial}{\partial y}(kg) = -\frac{B}{A}k \qquad (12)$$

If the system (11) possesses a cycle Γ inside the population quadrant ($x > 0$, $y > 0$) and R is the closure of the domain enclosed by the cycle, integration of (12) and use of Green's theorem gives

$$-\frac{B}{A}\int_R\int k(x,y)\,dx\,dy = \int_R\int \left\{\frac{\partial}{\partial x}(kf) + \frac{\partial}{\partial y}(kg)\right\} dx\,dy$$

$$= \int_{\Gamma} (k\,f\,dy - k\,g\,dx) = \int_0^T (k\,f\,g - k\,g\,f)\,dt = 0$$

where T is the period of the cycle Γ. However, the left hand side $\neq 0$ since $k(x, y)$ has no zeros in the domain R. The

assumption of the existence of a cycle is not tenable and thus there are no cycles for (11) as long as $A \neq 0$, $B \neq 0$.

D'Ancona - Volterra model (4)

This predator - prey model has the form

$$\frac{dx}{dt} = x(a - by) \quad , \quad \frac{dy}{dt} = y(-c + dx) \qquad (13)$$

where all the constants are positive.

Here $A = bd > 0$, $B = 0$ and the condition $B \neq 0$ does not hold. Every orbit of (13) is a cycle and integration of (13) gives the solution

$$a \log y - by + c \log x - dx = k_1 \qquad (14)$$

k_1 being a constant of integration.

In a quadratic model like the system (11) if $A = 0$ or $B = 0$ or both are zero, if there is one cycle in the system, then there are infinitely many cycles and they completely fill up a region in the population quadrant. Isolated cycles i.e. limit cycles do not exist at all.

A biological cycle persevering over a substantial length of time must be insensitive to the shocks and disturbances of the 'real world'. If a disturbance at time t_1 suddenly knocks a population point p off a cycle Γ into a nearby point q and then the disturbance stops, the orbit starting at q at time t_1 should move back towards Γ as time goes on. Such a cycle is ecologically stable (asymptotic orbital stability). It is likely that the only exact population cycles that exist in nature are those that are ecologically stable.

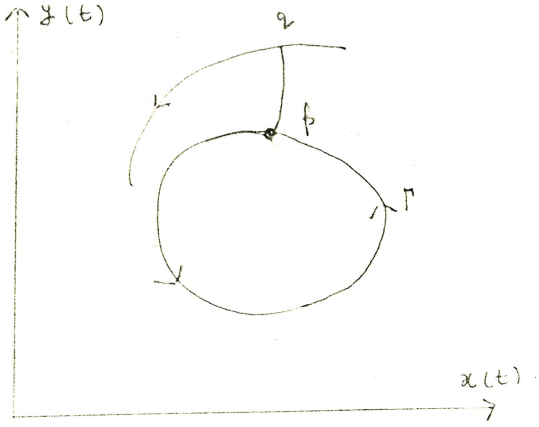


Figure 3
(ecologically stable cycle)

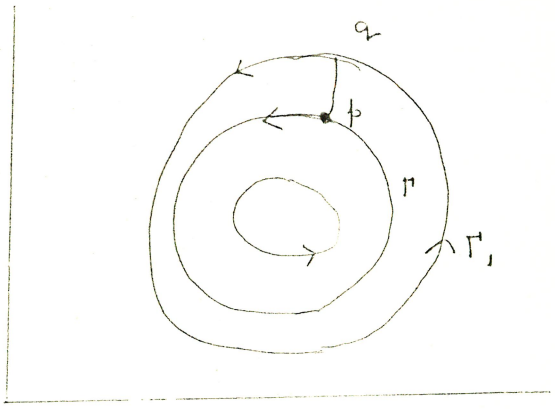


Figure 4
D'Ancona-Volterra cycles (not ecologically stable

The D'Ancona-Volterra cycles (eqns. (4), (6)) are not ecologically stable and a sudden shock may force a population point p off a cycle Γ and into a different cycle Γ_1. The population remains on Γ_1 unless another shock sends the population onto yet another cycle. A return to the original cycle Γ is unlikely. An ecologically stable cycle must be isolated with no other nearby cycles. Quadratic population models cannot have isolated cycles and do not possess ecologically stable cycles. The system

$$\frac{dx}{dt} = x(a-\alpha x-by) \ , \ \frac{dy}{dt} = y(-c + dx - \beta y) \qquad (15)$$

involves the overcrowding coeffients α, β both positive and small in comparison to a, b, c and d.

$$A = \alpha\beta + bd > 0, \ B = c\ \alpha\ \beta - (\alpha bc + \alpha\beta a + \beta ad) \qquad (16)$$

$B < 0$ if $\alpha\beta$ is sufficiently small. Then neither A nor B is zero and the system (15) has no cycles in the population quadrant. However, since α and β are both small, the system (15) is `close' to a D'Ancona-Volterra model. If we pick a population point (x_o,

y_o) 'near' the equilibrium point (c/d, a/b) of (13) and track the orbit through (x_o, y_o) once around (c/d, a/b), it returns precisely to (x_o, y_o) since (13) has cycles. However, if we begin at (x_o, y_o) and follow the orbit of (15) around one complete turn, the orbit misses (x_o, y_o) and slips past on the inside. This behaviour is repeated in the next.

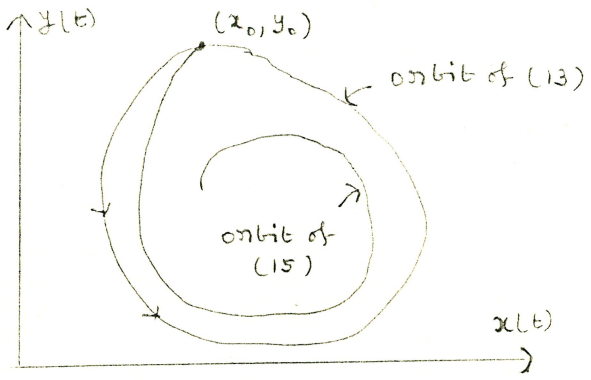


Figure 5. Orbits of (13) and (15)

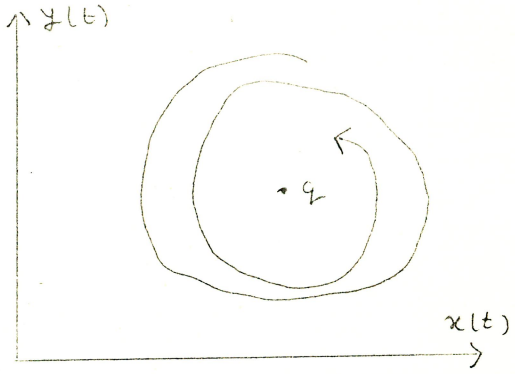


Figure 6. Spiral orbit of (15)

turn, again in the next and the successive turns of the orbit forma tightly coiled spiral which slowly works its way towards the equilibrium point. When α and β are small and (x_o, y_o) is close to the equilibrium point, the inward movement of the spiralling orbit is small and the orbit gives the appearance of a cycle to an observer. Thus (15) can still portray the predator - prey oscillations observed in the shark - fish community.

If the system (11) viz.,

$$\frac{dx}{dt} = (a_o + a_1x + a_2y)\,x, \quad \frac{dy}{dt} = (b_o + b_1x + b_2y)\,y \qquad (11)$$

has a cycle Γ defined by $x = x(t)$, $y = y(t)$ of period T in the population quadrant, the average populations $x = 1/T \int_o^T x(t)\,dt$, $y = 1/T \int_o^T y(t)\,dt$ are such that

$$a_o + a_1 \bar{x} + a_2 \bar{y} = 0, \quad b_o + b_1 \bar{x} + b_2 \bar{y} = 0.$$

$$\frac{1}{T}\int_0^T \frac{1}{x}\frac{dx}{dt}\,dt = \frac{1}{T}\int_0^T (a_o + a_1 x + a_2 y)\,dt = (a_o + a_1 \bar{x} + a_2 \bar{y})$$

and the left side expression $= \frac{1}{T}\log\frac{x(T)}{x(o)} = 0$

since $x(t)$ has period T. Similarly $b_o + b_1 \bar{x} + b_2 \bar{y} = 0$.

If the predator - prey model (15) with overcrowding has cycles and $\beta < b$ and both the species are harvested at a rate proportionate to their populations, the constants a and $-c$ in (15) are replaced by $a - h$ and $-c - h$ respectively. Also, we still have

$$a - h - \alpha \bar{x} - b \bar{y} = 0 = -c - h + d \bar{x} - \beta \bar{y}$$

$$\bar{x} = \frac{bc + a\beta + (b-\beta)h}{\alpha\beta + ab} > \frac{bc + a\beta}{\alpha\beta + ab}$$

$$\bar{y} = \frac{(a - h)d - \alpha(c+h)}{\alpha\beta + ab} < \frac{ad - \alpha c}{\alpha\beta + ab}$$

i.e. if the harvested model still has cycles, the average population of the prey species x along a cycle increases with harvesting but the average population of the predator species y decreases.

Biological cycles; the fivefold way

Cycles of darkness and light, of seasons and of the weather are well known. However, it is a curious phenomenon that there is a regular periodic variation in the populations of certain interacting species, which does not correlate with periodic external forces like darkness and light. These species are

involved in a predator - prey relationship..

e.g. the microcosm of a laboratory culture, the macrocosm of the Canadian coniferous forests (Canadian lynx (predator) and snowshoe hare (prey), budworm-larch tree cycle in the Swiss Alps).

The Canadian coniferous forests extend over a vast area and it is known for a long time that the populations of several species of birds, fish and mammals in this region vary periodically in a remarkably regular and predictable cycle of approximately ten years. The maximal amplitudes vary but the regularity of the cycle and the great differences between the numbers in the peak and in the lean years are striking. A number of explanations have been offered and one such is that the lynx (predator) and another species of flora or fauna form a closed interacting community whose internal reactions cause cycles in the numbers of both the species.

A two-species population model

Let x(t), y(t) denote the respective numbers of inviduals (at time t) which form a closed interacting community. In the time span t to t+Δt the changes in x and y are Δx and Δy. If B and M denote the birth rate and the mortality rate and the community is closed, there are no other forces that affect the relative rates of change of the populations and

$$\frac{1}{x}\frac{dx}{dt} = B_1(x, y) - M_1(x, y) = f(x, y) \qquad (17)$$

$$\frac{1}{y}\frac{dy}{dt} = B_2(x, y) - M_2(x, y) = g(x, y) \qquad (17)$$

The relative rates of change viz., f and g are continuously differentiable in x and y for $x \geq 0$, $y \geq 0$. The system (17) is a specific type of the more general planar autonomous differential equation system

$$\frac{dx}{dt} = F(x, y) \quad , \quad \frac{dy}{dt} = G(x, y) \qquad (18)$$

where F and G are continuously differentiable functions of x, y in a region R of the x-y plane. The region R may be the entire plane. A pair of functions x(t), y(t) is a solution of (18) if

(i) x(t), y(t) are defined on a common open interval I : $a < t < b$ (a may be $-\infty$ and b may be $+\infty$);

(ii) x(t), y(t) satisfy (18) for every $t \in I$

(iii) there are no functions $x^*(t)$, $y^*(t)$ defined on $I^* \supset I$ that coincide with x(t), y(t) on I and satisfy (18) for all $t \in I^*$.

Condition (iii) means that we seek solutions of (18) that are maximally extended in time. Each solution of (18) defines a curve Γ in the x-y plane, the so-called orbit of the solution. We assume that (a) through each point of R there passes an orbit, (b) no two orbits intersect unless they coincide, (c) each orbit is either a non-self intersecting smooth curve, or a simple smooth closed curve, or a single point.

$$\text{e.g. } \frac{dx}{dt} = y \quad , \quad \frac{dy}{dt} = -x \qquad (19)$$

Solution S1: $x_1(t) = \cos t$, $y_1(t) = -\sin t$ $-\infty < t < \infty$

Solution S2: $x_2(t) = \sin t$, $y_2(t) \cos t$ $-\infty < t < \infty$.

The two solutions are distinct but define the same orbit Γ: $x^2 + y^2 = 1$, oriented in the clockwise manner.

$$x(t) = c_1 \cos t + c_2 \sin t$$

$$y(t) = -c_1 \sin t + c_2 \cos t$$

give the general solution of the system and the orbit is the system of concentric circles

$$x^2 + y^2 = c_1^2 + c_2^2$$

When both c_1 and c_2 are zero the critical point $x = 0$, $y = 0$ is a solution of (19) and all other solutions are circles, i.e. cycles. Each cycle (i.e. whatever c_1 and c_2 may be barring the case $c_1 = c_2 = 0$) has the period 2π.

A solution of (18) viz.,

$$\frac{dx}{dt} = F(x, y) \ , \qquad \frac{dy}{dt} = G(x, y) \tag{18}$$

is periodic (cyclic) if a positive number T exists such that for all t,

$$x(t + T) = x(t) \ , \quad y(t + T) = y(t) \tag{20}$$

and the smallest number T is the period of the cycle. The orbit of the system corresponding to the cycle is is a simple closed curve (i.e. a non-intersecting closed curve). Any orbit which is a simple closed curve in the phase plane corresponds to a periodic solution. A formal proof of this seemingly obvious assertion is tedious.

A point orbit is a critical point and is a fixed point of the system. The point (x_o, y_o) is critical for the system (18) and only if

$$F(x_o, y_o) = 0 \quad , \quad G(x_o, y_o) = 0$$

Inside every cycle there is at least one critical point. Proving this statement is not easy.

The system in the example (19) has one critical point and an infinity of cycles all of which have the same period = 2π.

<u>An example with only one critical point and only one cycle</u>

$$\frac{dr}{dt} = r(1-r^2) \ , \ r \geq 0 \ ; \quad \frac{d\Theta}{dt} = 1 \qquad (21)$$

(r, Θ) are plane polar coordinates. In terms of rectangular Cartesian coordinates (x, y) with $x = r\cos\Theta$, $y = r\sin\Theta$ the system (20) becomes

$$\frac{dx}{dt} = (1-x^2-y^2)x-y, \quad \frac{dy}{dt} = (1-x^2-y^2)y+x \qquad (21a)$$

The origin $(x = 0, y = 0)$ or $r = 0$ is a critical point of the system and the circle $x^2 + y^2 = 1$ $(r = 1, \Theta(t) = t + \Theta_o)$ is the only cycle with the parametric equations

$$x(t) = \cos(t+\Theta_o) \ , \ y(t) = \sin(t+\Theta_o) \qquad (21b)$$

and with period = 2π.

The system (20) has the solution

$$r(t) = r_o \{r_o^2 + (1-r_o^2)\, e^{-2t}\}^{-1/2}, \ r_o \geq 0$$

$$\Theta(t) = t + \Theta_o \quad (\Theta_o \text{ arbitrary} \qquad (21c)$$

with $r = r_o$, $\Theta = \Theta_o$ at $t = 0$. If $r_o = 1$, $r(t) = 1$ (cycle) and if

$r_o = 0$, $r = 0$ (fixed point or critical point). If $0 < r_o < 1$, $r(t)$ increases monotonically to 1 from r_o as t increases from 0 to ∞. If $r_o > 1$, $r(t)$ decreases from r_o to 1 as t increases from 0 to ∞. Thus whether r_o lies between 0 and 1 or is greater than 1, $r(t) \to r(\infty) = 1$ as t increases to ∞. The orbit thus approaches the cycle $r(t) = 1$ which is an equilibrium orbit. in the example (19) every circle of the concentric system $x^2 + y^2 = c_1^2 + c_2^2$ is a cycle and hence an equilibrium orbit.

Critical points and orbits that are cycles are called equilibria or equilibrium orbits.

The dynamics of a pair of interacting species is described by a system of differential equations of the form (c.f. (17)).

$$\frac{dx}{dt} = x\, f(x, y) \quad , \quad \frac{dy}{dt} = y\, g(x, y) \qquad (17a)$$

and neither of the examples in (19), (20) has this form of equations. However, both these examples possess cycles and a model of the lynx-hare interaction can be expected to possess cycles. The orbits for the systems (19), (20) may thus be possible diagrams of the lynx-hare orbits.

Determination of the equilibrium orbits of a system like (18) viz.,

$$\frac{dx}{dt} = F(x, y) \quad , \quad \frac{dy}{dt} = G(x, y) \qquad (18)$$

thus consists in the detection of critical points and the presence of cycles among the orbits of the system. Critical points are determined by solving the pair of equations $F(x, y) =$

0, G (x, y) = 0. To find the orbits of the system and then detect the presence of any cycles, we need the solution of the plane autonomous system (18) and this is not always a simple task. It is therefore desirable to develop a criterion for the existence of a cycle for the system without the need for prior solution of the system of differential equations.

A solution of (18) is a pair of functions $x = x(t)$, $y = y(t)$ that satisfy (18) for all time t in some maximal time interval. If $(x(0), y(0))$ is the initial point P of the solution and Γ is the corresponding orbit, we may denote the solution by Z (t, P) with Γ as its orbit.

The subset of Γ with $\{ Z(t, P) ; t \geq 0 \}$ is the positive semi-orbit of the system from P and is denoted by Γ_P^+. We may define Γ_P^- in a similar way.

If Γ_P^+ is a positive semi-orbit of the system (18), the omega limit set $\omega(\Gamma_P^+)$ is the set of all Q such that for some increasing sequence $\{ t_n \}$ of time diverging to ∞, $Z(t_n) \to Q$. The alpha limit set $\alpha(\Gamma_P^-)$ is defined similarly with the sequence $\{t_n\}$ decreasing as n increases and diverging to $-\infty$.

Clearly, if Γ is a critical point of the system, $\alpha(\Gamma) = (\Gamma) = \Gamma$. Liexwise, if Γ is a periodic orbit, $\alpha(\Gamma) = \omega(\Gamma) = \Gamma$.

The limit sets $\omega(\Gamma_P^+)$, $\alpha(\Gamma_P^-)$ are independent of the choice of the point P on Γ.

Let Γ_P^+ be a bounded positive semi-orbit of the system (18) and let $Q \varepsilon \Gamma$. If $(x(t), y(t))$ is a solution of (18) for all t, a

$< t < b$ and $(x_c(t), y_c(t)) = (x(t-c), y(t-c)$, the latter is a solution of the system (18) for all t such that $a < t - c < b$, i.e. $a+c < t < b+c$ $(x(t), y(t))$ and $(x_c(t), y_c(t))$ define one and the same orbit with t or t-c as the parameter. Hence $\omega(\Gamma_P^+) = (\Gamma_P^+)$ on identifying $P = (x(o), y(o))$ and $Q = (x(c), y(c))$ both of which lie on the orbit Γ. We may see that $\alpha(\Gamma_P^-) = \alpha(\Gamma_Q^-)$ in a similar way.

Example

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = -y \tag{22}$$

Solution : $x(t) = x(o)\, e^{t}$, $y(t) = y(o)\, e^{-t}$.

The orbit through (0, 1) is $x(t) = 0$, $y(t) = e^{-t}$. Let this orbit be Γ_1. As $t \to \infty$, $y(t) \to 0$ and $\omega(\Gamma_1) = (0,0)$. As $t \to -\infty$, $y(t) \to +\infty$ and $(0, +\infty)$ is not a point on Γ_1. $\alpha(\Gamma_1)$ is thus empty. Orbit Γ_2 : $x = e^{t}$, $y = 0$. $\omega(\Gamma_2)$ is empty and $\alpha(\Gamma_2) = (0, 0)$. For the orbit P_3 $(x = e^{t}, y = e^{-t})$ both $\omega(\Gamma_3)$ and $\alpha(\Gamma_3)$ are empty sets.

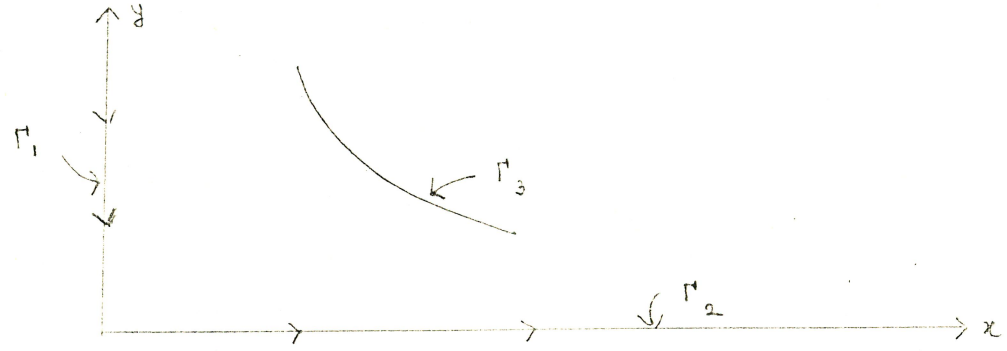


Figure 7 : Orbits and limit sets of the system (21)

Example

$$\frac{dr}{dt} = r(1-r^2), \quad r \geq 0 \qquad \frac{d\Theta}{dt} = 1 \tag{21}$$

$$r(t) = r_o \{ r_o^2 + (1-r_o^2)\, e^{-2t} \}^{-1/2},\ r_o \geq 0 \qquad (21c)$$

$$\Theta(t) = t + \Theta_o$$

$\Gamma_1 : 0 < r_o < 1 : r \to 1$ from r_o as $t \to +\infty$

$\Gamma_2 : r_o > 1 : r \to 1$ from r_o as $t \to +\infty$

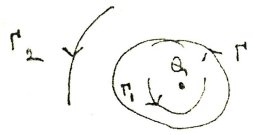
If $r_o = 0$, $r(t) = 0$ and $Q : (r = 0)$. As t $+\infty$, r 0 and as $t \to -\infty$, $r \to 0$.

If $r_o = 1$, $r(t) = 1$ and $\Gamma : r(t) = 1$. Then

$\omega(\Gamma) = \alpha(\Gamma) = \Gamma$.

If $0 < r_o < 1$ nd $t \uparrow \infty$, $\omega(\Gamma_1) = \Gamma$, $\alpha(\Gamma_1) = Q$. If $r_o > 1$, $\omega(\Gamma_2) = \Gamma$, $\alpha(\Gamma_2)$ is empty.

If $r_o = 0$, $r(t) = 0$ and $Q : (r = 0)$. As $t \to +\infty$. $r \to 0$.



Γ: $r(t) = 1$. Q : $r(t) = 0$. Γ_1 Γ as t $+\infty$

Γ_2 $r(t) = 1$ i.e. Γ as t ∞, Figure 8.

<u>Example</u>

$$\frac{dr}{dt} = r(1-r^2), \quad \frac{d\Theta}{dt} = r^2 \sin^2\Theta + (r^2-1)^2 \qquad (23)$$

Critical points $(r = 1, \Theta = 0)$, $(r = 1, \Theta = \pi)$, $r = 0$

Let $(r = 1, \Theta = \pi)$ be P_1, $(r = 1, \Theta = 0)$ be P_2 , $r = 0$ be P_3.

$\frac{d\Theta}{dt} > 0$ except at the critical points P_1, P_2, P_3.

$\frac{dr}{dt} > 0$ if $0 < r < 1$, $\frac{dr}{dt} = 0$ if $r = 1$ and

$\frac{dr}{dt} < 0$ if $r > 1$

Γ_1 : $r = 1$, $\pi < \Theta < 2\pi$

Γ_2 : $r = 1$, $0 < \Theta < \pi$

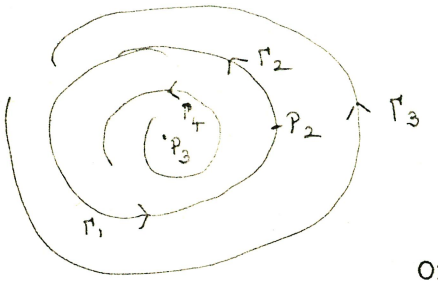
Γ_3 : $r > 1$, $\frac{dr}{dt} < 0$, r decreases to $r = 1$

Γ_4 : $0 < r < 1$, $\frac{dr}{dt} > 0$, r increases to $r = 1$

$(\Gamma_3) = \Gamma_1 \cup \Gamma_2 \cup \{ P_1, P_2 \}$

$\alpha(\Gamma_3)$ is empty.

$(\Gamma_2) = \{ P_1 \} = \alpha\{ \Gamma_1 \}$, $\alpha\{\Gamma_2\} = \{P_2\} = \{\Gamma_1\}$



Orbits of (23)

Figure 9

$$\frac{dr}{dt} = r(1-r^2) \quad \Rightarrow \quad r(t) = r_o\{r_o^2 + (1-r_o^2)e^{-2t}\}^{-1/2}, \; r_o \geq 0$$

If $r_o > 1$, we have

$$1 < r_o^2 + (1-r_o^2)e^{-2t} < r_o^2$$

and

$$r_o > r_o\{r_o^2 + (1-r_o^2)e^{-2t}\}^{-1/2} > 1$$

If $0 < r_o < 1$, we have

$$r_o < r_o\{r_o^2 + (1-r_o^2)\, e^{-2t}\}^{-1/2} < 1$$

and hence $r(t) < 1$ and $r(t)$ increases to limit 1 as $t \uparrow \infty$. Hence the orbit Γ_4 of the system (23) in figure 9. Orbits Γ_1 and Γ_2 are both subsets of $r(t) = 1$. Since $\omega(\Gamma_3) = \Gamma_1 \cup \Gamma_2 \cup \{ P_1,$

$P_2\}$, the ω - limit set is not a single critical point nor even a cycle but is a connected union of orbits with a finite number (exactly two in this example) of critical points; further, $\alpha(\Gamma_1) = \{ P_1 \}$, $\omega\{\Gamma_1\} = \{P_2\}$ $\alpha\{\Gamma_2\} = \{P_2\}$, $\{\Gamma_2\} = \{P_1\}$; and each limit set of each of the orbits (Γ_1, Γ_2) is a critical point of the system. $\omega(\Gamma_3) = \Gamma_1 \cup \Gamma_2 \cup \{P_1, P_2\}$ is an instance of a cycle graph consisting of a connected union of orbits and with the other properties mentioned above.

Theorem of poincare and Bendixon - (The fivefold way)

Let Γ^+ be a bounded (i.e. Γ^+ lies in a rectangle) positive semi-orbit of the system (18), viz.,

$$\frac{dx}{dt} = F(x, y) \quad , \quad \frac{dy}{dt} = G(x,y) \qquad (18)$$

and let there be only a finite number of critical points of the system. Then one and exactly one of the following five alternatives will be valid.

(a) Γ is a critical point;

(b) Γ is positively asymptotic to a critical point;

(c) Γ is a cycle;

(d) Γ is positively asymptotic to a cycle;

(e) Γ is positively asymptotic to a cycle graph.

(a) . Γ ; (b) Γ (c)

(d) ; (e) Γ

A similar result holds for $\alpha(\Gamma)$ if Γ^{-1} is bounded. If the possibilities for $\omega(\Gamma)$ and $\alpha(\Gamma)$ are combined, there arise eleven alternatives for the limit sets of a bounded orbit.

List of the eleven possible alternatives of alpha and omega limit sets or orbit lying in a bounded region containing only a finite number of critical points

(i) Γ is a critical point $\cdot\ \Gamma$

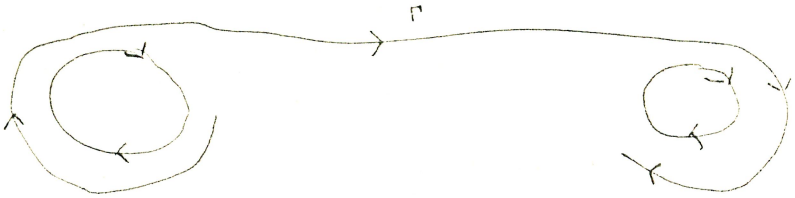
(ii) Γ is a cycle

(iii) Γ is positively asymptotic to a critical point

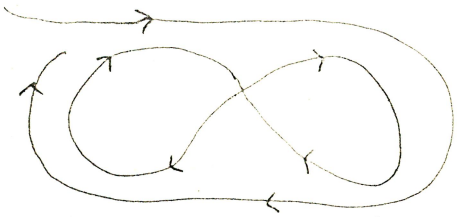
(iv) Γ is positively asymptotic to a cycle

(v) Γ is negatively asymptotic to a cycle

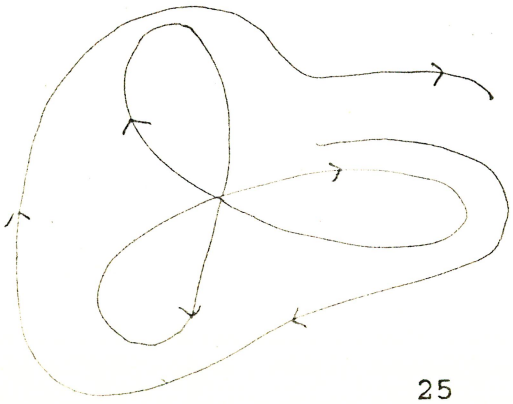
(vi) Γ is positively asymptotic to a cycle and negatively asymptotic to another cycle



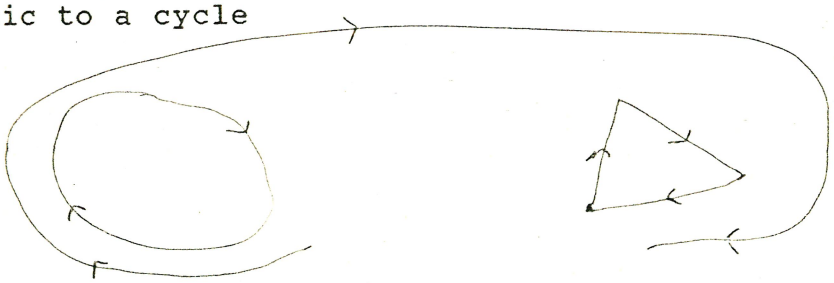
(vii) Γ is positively asymptotic to a cycle graph



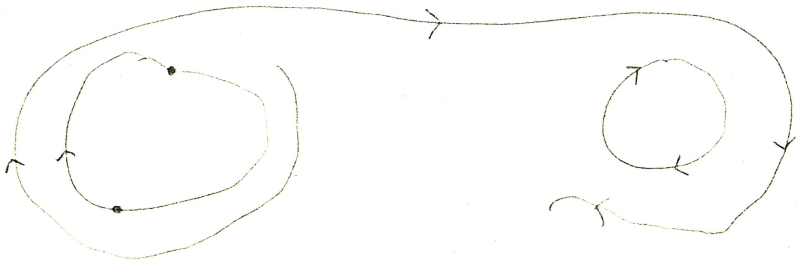
(viii) Γ is negatively asymptotic to a cycle graph



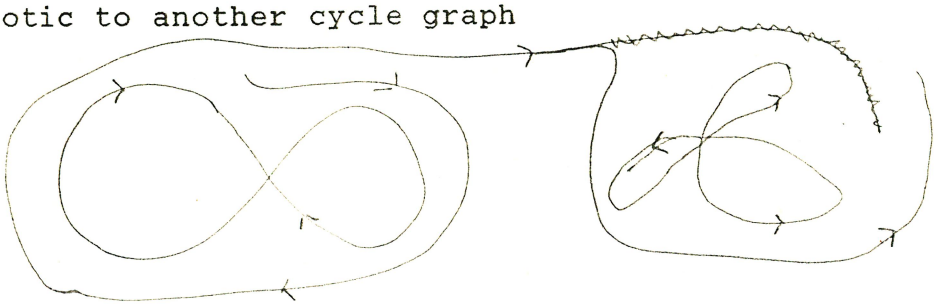
(ix) Γ is positively asymptotic to a cycle graph and negatively asymptotic to a cycle



(x) Γ is positively asymptotic to a cycle and negatively asymptotic to a cycle graph



(xi) Γ is positively asymptotic to a cycle graph and negatively asymptotic to another cycle graph



The fivefold way (theorem of Poincaré and Bendixon) shows that we can conclude the existence of a cycle if there is a positive semi-orbit Γ^+ which is bounded and whose positive limit set $\omega(\Gamma)$ contains no critical points. In such a case $\omega(\Gamma)$ is itself a cycle.

Repellent critical point. A critical point P of the autonomous system (18) is repellent if there is a circular region D centred at P such that

(i) for all $Q \in D$, $Z(t, Q) \to P$ as $t \to -\infty$

(ii) for all $Q \in D$ and all sequences $\{ t_n \}$ with $t_n \to \infty$ and $Z(t_n, Q)$ defined, $Z(t_n, Q) \not\to P$.

A simple algebraic test for repellence

Let (x_o, y_o) be a critical point of (18) and let a, b, c, d be the constants defined below.

$$a = \left(\frac{\partial F}{\partial x} \right)_{(x_o, y_o)}, \quad b = \left(\frac{\partial F}{\partial y} \right)_{(x_o, y_o)}, \quad c = \left(\frac{\partial G}{\partial x} \right)_{(x_o, y_o)},$$

$$d = \left(\frac{\partial G}{\partial y} \right)_{(x_o, y_o)}$$

Then (x_o, y_o) is a repellent critical point if $a+d > 0$ and $ad-bc > 0$. In other words the eigenvalues of the Jacobian matrix $\partial(F, G)/\partial(x, y)$ at the critical point have positive real parts.

The fivefold way and the repellence test enable us to deduce the existence of a cycle without the knowledge of the analytical solutions of the differential equations system.

Example

$$\frac{dr}{dt} = r(1-r^2), \; r \geq 0 \; ; \quad \frac{d\Theta}{dt} = 1 \qquad (21)$$

Let R be the circular region $0 \leq r \leq 2$ with the only critical point $r = 0$ at the centre. The system (21) can be transformed into the system

$$\frac{dx}{dt} = (1-x^2-y^2)x-y, \quad \frac{dy}{dt} = (1-x^2-y^2)\, y + x \qquad (21a)$$

with $x = r\cos\Theta$, $y = r\sin\Theta$. It is seen that

$$a = 1, \; b = -1, \; c = 1, \; d = 1$$

and $a+d = 2 > 0$, $ad - bc = 2 > 0$. By the algebraic test

mentioned above, the critical point (0, 0) is repellent. At a point P on the boundary of R(i.e. r = 2), dr/dt = -6 < 0. Γ_P^+ enters R through P and can never go out of R. The fivefold way applies to Γ^+_P. The repellent critical point cannot belong to $\omega(\Gamma_P^+)$ and R has no other critical point. Hence, $\omega(\Gamma_P^+)$ is a cycle r = 1 is a cycle as seen earlier.

Predator - Prey Models

The system (17) viz.,

$$\frac{dx}{dt} = x\, f(x, y) \;,\quad \frac{dy}{dt} = y\, g\,(x, y) \qquad (17)$$

can be used to model two species interactions.

Prey species : x(hare) ; predator : y (lynx)

f and g are continuously differentiable throughout the population quadrant $x \geq 0$, $y \geq 0$.

Conditions on f and g that reflect p.ossible predator - prey interactions.

(i) $\frac{\partial f}{\partial y} < 0$ (increase in the lynx population adversely affects the hare)

(ii) $\frac{\partial g}{\partial x} > 0$ (increase in the hare population favourably affects the lynx)

(iii) $\frac{\partial f}{\partial x} < 0$; (iv) $\frac{\partial g}{\partial y} < 0$

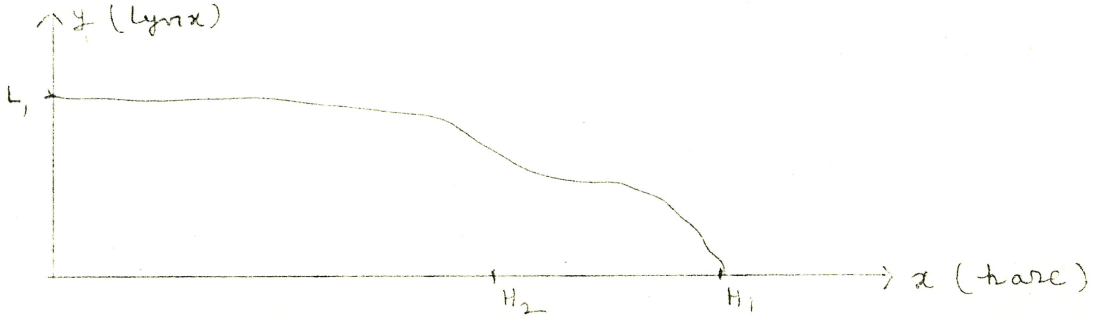
(v) $f(0, L_1) = 0$ for some $L_1 > 0$; beyond the threshold population L_1 of lynx, the hare population decreases.

(vi) $f(H_1, 0) = 0$ for some $H_1 > 0$; H_1 is the carrying capacity of the system for the hare in the absence of the lynx.

(vii) $g(H_2, 0) = 0 \quad (H_2 > 0)$; H_2 is a lower bound on the numbers of hare needed to support the lynx community.

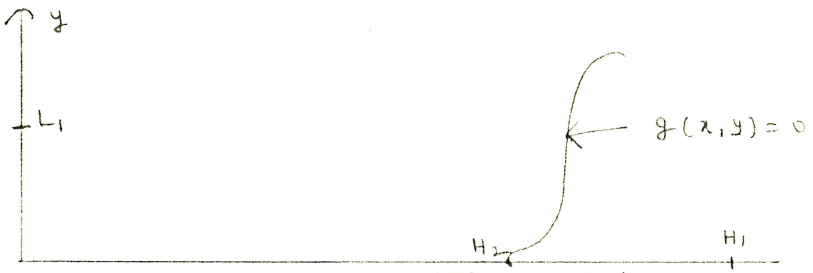
(viii) $H_1 > H_2$

(ix) $f(x, y) = 0$ (hare isocline) has the following shape.



(Figure 10)

(x) $g(x, y) = 0$ (lynx isocline) has the following shape.



(Figure 11)

$g(x, y) = 0$ at $x = H_2$ and as x increases g increases vide condition (ii). The isoclincs of hare and lynx intersect exactly once in the first quadrant and thus the system (17) has exactly one critical point $Q(x_o, y_o)$ in the population quadrant.

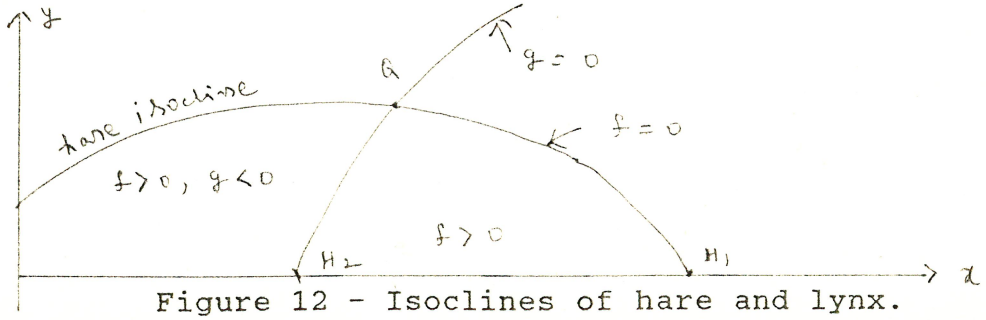


Figure 12 - Isoclines of hare and lynx.

The system (17) is the same as the system (18) with $F(x, y) = x f(x, y)$, $G(x, y) = y g(x, y)$

$$a = \frac{\partial F}{\partial x} \text{ at } (x_o, y_o) = x_o \frac{\partial f}{\partial x} (x_o, y_o) < 0$$

$$d = \frac{\partial G}{\partial y} \text{ at } (x_o, y_o) = y_o \frac{\partial y}{\partial y} (x_o, y_o) < 0$$

$a + d < 0$. Hence the critical point $Q(x_o, y_o)$ is not repellent.

We are looking for cycles or periodic solutions of the system (17).

Kolmogorov's cycle theorem. Let the functions f and g of (17) satisfy the conditions in the population quadrant.

$$\text{(i) } \frac{\partial f}{\partial y} < 0 \text{ , (ii) } \frac{\partial g}{\partial x} > 0 \text{ , (iii) } \frac{\partial f}{\partial x}\Big|_{L=0,\ H \text{ small}} > 0 \text{ ,}$$

$$\text{(iv) } \frac{\partial g}{\partial y} < 0 \text{ , (v) } f(0, L_1) = 0 \ (L_1 > 0),$$

(vi) $f(H_1, 0) = 0$ $(H_1 > 0)$, (vii) $g(H_2, 0) = 0$, $H_2 > 0$, (viii) $H_1 > H_2$ (ix) The isoclines of hare (H) and lynx (L) have the shapes shown in Figure (12) and the critical point Q lies on the rising part of the hare isocline.

Then a cycle exists inside the population quadrant.

x x x x x x

Do hares eat the lynx?

Consider the model

$$\frac{dx}{dt} = x \{\alpha^2 - \beta^2 (x-\gamma^2)^2 - \delta^2 y\}$$

$$\frac{dy}{dt} = y\{-a^2+b^2-c^2y\} \qquad (25)$$

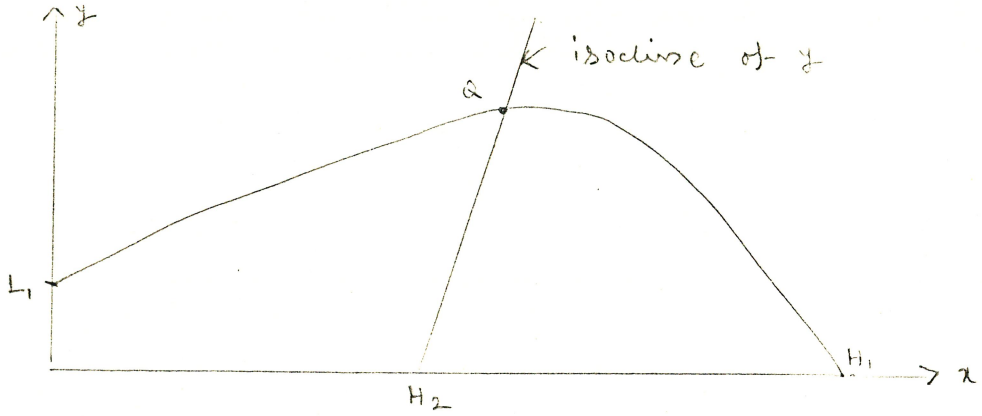
$$\frac{\partial f}{\partial y} = -\delta^2 < 0, \quad \frac{\partial g}{\partial x} = b^2 > 0, \quad \frac{\partial f}{\partial x} = -2\beta^2(x-\gamma^2)$$

and if x is small, $\left.\frac{\partial f}{\partial x}\right|_{y=o,\ x\ small} = 2\beta^2(\gamma^2-x)\Big|_{x\ small} > 0,$

$\frac{\partial f}{\partial y} = -c^2 < 0$, $f(0, L_1) = \alpha^2 - \beta^2\gamma^4 - \delta^2 L_1 = 0$ if

$L_1 = \frac{\alpha^2 - \beta^2\gamma^4}{\delta^2}$ and $L_1 > 0$ by suitable choice of $\alpha, \beta, \gamma, \delta$.

$f(H_1, 0) = \alpha^2 - \beta^2 (H_1 - \gamma^2)^2 = 0$ if $H_1 = \gamma^2 + \alpha/\beta$ and $H_1 > 0$. $g(H_2, 0) = -a^2 + b^2 H_2 = 0$ if $H_2 = a^2/b^2$, $H_1 > H_2$ if $\gamma^2 + \alpha/\beta > a^2/b^2$. The isoclines of x and y (i.e. the curves $f(x, y) = 0$, $g(x, y) = 0$) have the shapes shown in the figure below.



Isocline of x is the parabola $y = \alpha^2/\delta^2 - \beta^2/\delta^2 (x- \ ^2)^2$ and the isocline of y is the straight line $b^2x - c^2y - a^2 = 0$.

The model (25) meets all the conditions of Kolmogorov's theoremif $\alpha^2 > \beta^2\gamma^4$, $\delta > 0$ and $\alpha/\beta + \gamma^2 > a^2/b^2$. By K's theorem, the system has a cycle. From (25), we have

$$\frac{dx}{dt} = -\beta^2 x^3 + 2\beta^2\gamma^2 x^2 + (\alpha^2-\beta^2\gamma^4)x - \delta^2 xy$$

If $\alpha^2 > \beta^2\gamma^4 > 0$, $\delta > 0$ and x and y are small, the quadratic term $2\beta^2\gamma^2x^2$ and the linear term $(\alpha^2-\beta^2\gamma^4)x$ dominate over the terms x^3 and xy and x (i.e., the hare population) grows rapidly. When x or y is sufficiently large, the terms in x^3 and xy dominate and the growth rate slackens and may even become negative.

In a certain model study based on `best fit' choice of constants for the lynx-hare communities the constants in (25) are changed by the exchanges of $-\delta^2$ with δ^2 and b^2 with $-b^2$. After this change, an encounter between a hare and a lynx increases the hare population x and lowers the lynx population y and then we have the biological miracle of the hare eating the lynx. This is visualisable by means of a disease carried by the hares which is fatal to lynx but not to the hares.

How many cycles? Hilbert's sixteenth problem

B. Van der Pol (Phil. Mag., Vol. 6, Ser. 7., (1928), pp. 763-765) has grouped a most varied class of phenomena under a single category of systems that exhibit cycles or periodic solutions, even though no `external periodic force' causes such oscillations. "..... the aeolian harp, a pneumatic hammer, the scratching noise of a knife on a plate, the waving of a flag in the wind, the humming noise sometimes made by a water-tap, the squeaking of a door, the tetrode multivibrator, ... the intermittent discharge of a condensor through a neon tube, the

periodic reoccurrence of epidemics and of economical crises, the periodic density of an even number of species of animals living together and the one species serving as food for the other, the sleeping of flowers, the periodic recurrence of showers behind a depression, the shivering from cold, menstruation, and finally, the beating of the heart".

Most of these systems can be modelled more or less accurately by the pair of differential equations

$$\frac{dx}{dt} = F(x, y) \quad , \quad \frac{dy}{dt} = G(x, y) \tag{18}$$

e.g., (i) x = voltage across the condensor (capacitor) of a neon tube, y = rate at which the voltage changes.
(ii) x = population density of a predator, y = population density of the prey.

We are interested in the cycles (periodic solutions) of the above autonomous system.

As noticed earlier we are interested in the solutions that are maximally extended in time. We assume that no two orbits of the system intersect unless they coincide and that an orbit is either a non-self intersecting curve, a simple closed curve, or a single point (critical point). A nonconstant solution x = x(t), y = y(t) is a cycle or periodic solution if a positive number T exists such that

$$x(t + T) = x(t), \quad y(t+T) = y(t)$$

for all t and then the orbit is a cycle. Cycles which appear are internally generated by the interactions of the x matter and the

y matter, whatever the matter(s) may be. Since the system is autonomous, there is no possibility or provision for some external, periodically varying source.

The system $\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x$

has infinitely many cycles and just one critical point at (0, 0). The system

$$\frac{dr}{dt} = r(1-r^2), \; r \geq 0, \quad \frac{d\theta}{dt} = 1$$

has just one critical point at r = 0 and just one cycle (r = 1). All orbits 'near' to r = 1 approach the cycle as time advances and hence the cycle is asymptotically stable. The cycle is 'isolated'.

The system $\frac{dx}{dt} = 1 - x, \quad \frac{dy}{dt} = -2y$

has a critical point at (1, 0) and there are no cycles for this system. The general solution is

$$x(t) = 1 - (1-x_o)\, e^{-t}, \; y(t) = y_o\, e^{-2t}.$$

The system $\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -4x + \cos t$

is non-autonomous and is driven by the external periodic force x = 0, y = cost

$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3} \cos t$$

$$y = -2c_1 \sin 2t + 2c_2 \cos 2t - \frac{1}{3} \sin t$$

and the system has a cycle with period 2π which is also the

period of the external force.

The system

$$\frac{dr}{dt} = r(1-r^2)\ (4-r^2)\ ..\ (N^2-r^2),\ r \geq 0\ (N = a$$

positive integer), $\quad \frac{d\Theta}{dt} = 1 \qquad (26)$

is the same as the system

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y)$$

with $F(x, y) = x(1-x^2-y^2)(4-x^2-y^2)..(N^2-x^2-y^2)-y$

$$G(x, y) = y(1-x^2-y^2)\ (4-x^2-y^2)..\ (N^2-x^2-y^2) + x \qquad (27)$$

F and G are polynomials in x and y of degrees 2N+1 each. There are N isolated cycles, viz.,

$r = 1,\ r = 2,\ \ldots,\ r = N$

which forms a family of concentric circles.

A general question is the determination of the number of isolated cycles for the autonomous system

$$\frac{dx}{dt} = F\ , \quad \frac{dy}{dt} = G$$

David Hilbert (1900) - The second part of the sixteenth among the 23 problems posed by Hilbert at the Second International Congress of Mathematicians, Paris, is to find the maximum number and position of Poincare's boundary cycles (limit cycles) (i.e., isolated cycles) for the first order, first degree differential equation

$$\frac{dy}{dx} = \frac{F(x,y)}{G(x,y)} \qquad (28)$$

where F, G are rational integral functions (polynomials) of the n-th degree in x, y.

Let P_n = the set of all real polynomials of degree $\leq n$. Clearly $P_{n-1} \subset P_n$ (n = 1, 2,...) and let

H(n) = supremum $_{(F,G)\ \varepsilon\ P_n}$ { number of isolated cycles of the autonomous system}

Clearly $H(n-1) \leq H(n)$. In the above example with N isolated cycles, n = 2N+1 and hence it is obvious that

$$H(n) \geq \left[\frac{n-1}{2}\right] \qquad (29)$$

Solution of the system dx/dt = α, dy/dt = β where α, β are constants shows that H(0) = 0.

Likewise, solution of $\frac{dx}{dt} = a_o + a_1x + a_2y$, $\frac{dy}{dt} = b_o + b_1 x + b_2 y$ (30)

shows that H(1) = 0.

Chin (1958, 1966) has given the example

$$\frac{dx}{dt} = -1-2y + x^2 + xy + y^2, \quad \frac{dy}{dt} = 2x - x^2 \qquad (31)$$

and it is seen easily that $x^2 + y^2 = 1$ defines a cycle for this system.

$x\frac{dx}{dt} + y\frac{dy}{dt} = x\,(x^2 + y^2 - 1)$ and this is zero on the circle $x^2 + y^2 = 1$. This is an isolated cycle but proving this is difficult. This example shows that $H(2) \geq 1$ and thus we see that

in the sequence H(0), H(2), H(3),....

H(2) is the first non-trivial element. Determination of the value of H(2) or of H(n) for any positive integer n is a non-trivial problem. Attempts to determine H(2) had a checkered course and its value is still not known. Guesses like H(2) = 3 and

$$H(n) = \leq \begin{cases} \frac{1}{2}(6n - 7n^2 - 11n + 6) & (n = 3, 5, 7, \ldots) \\ \frac{1}{2}(6n^3 - 7n^2 + n + 4) & (n = 4, 6, 8, \ldots) \end{cases} \qquad (32)$$

have turned out to be incorrect.

Quadratic systems have been identified with four isolated cycles (1979) and hence it may be asserted that H(2) ≥ 4. The system

$$\begin{aligned} \frac{dx}{dt} &= -\delta_2 x - y - 3x^2 + (1-\delta_1)\, xy + y^2 \\ \frac{dy}{dt} &= x + \frac{2}{9} x^2 - 3xy \end{aligned} \qquad (33)$$

(δ_1, δ_2 are small positive numbers).

(Chen, Wang (1979)) has four isolated cycles; three of them enclose the origin which is a critical point and a fourth cycle encloses the critical point (0, 1) whether the system has other isolated cycles is not clear.

The main results known regarding <u>quadratic systems</u> are that

(a) each cycle bounds a convex region containing a single critical point

(b) cycles enclosing a common critical point have a common orientation as t increases and that those enclosing distinct critical points are oppositely oriented.

(a) settles the problem of the possible shape of a cycle in a quadratic system and (b) implies that a quadratic system has at most two 'eyes', i.e. at most two critical points each of which is enclosed by one or more isolated cycles.

All that is known regarding H(n) for n > 2 is that H(3) ≥ 5 and

$$H(n) \geq \begin{cases} \frac{1}{2}(n^2 + 5n - 14) & (n = 6, 8, 10, \ldots) \\ \frac{1}{2}(n^2 + 5n - 26) & (n = 7, 9, 11, \ldots) \end{cases} \qquad (34)$$

The lower bounds here are distinctly better than the bound

$$H(n) \geq \left[\frac{n-1}{2}\right].$$

The determination of H(2), H(3), ... seems both hard and deep.

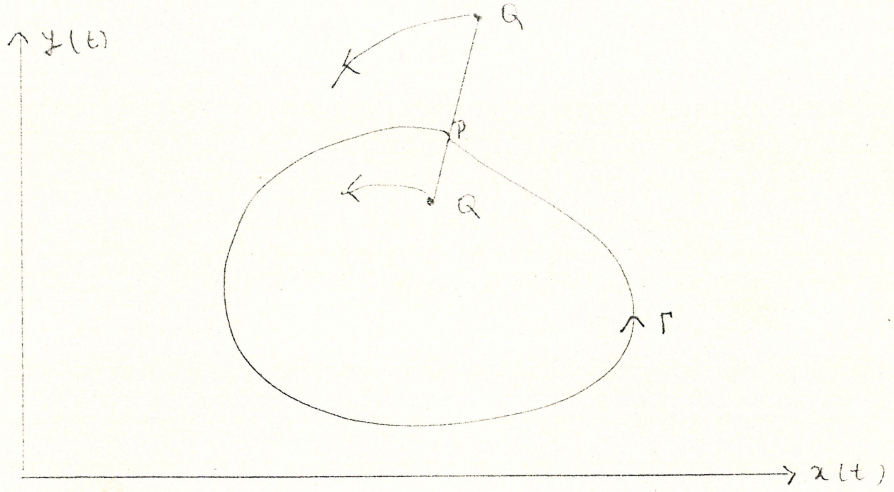
x x x x x x

The system

$$\frac{dx}{dt} = x\, f(x, y) \quad , \quad \frac{dy}{dt} = y\, g(x, y) \qquad (17)$$

and the isolated cycles that the system may possess are of concern in all the study of the interaction of biological species. Any cycles that may be present are to be internally

produced and not due to external periodic forces. i.e. we are looking for `natural' cycles of the system. A `natural' cycle persevering over a long time span must be insensitive to the shocks and disturbances of the `real world'. If a disturbance at time t, suddenly knocks a population density point $P_1 = (x_1, y_1)$ off a cycle Γ and onto a nearby point Q and if the disturbance ceases, will the orbit through Q more back towards Γ as time goes on. If it moves back towards Γ, the cycle is ecologically stable. In



ecologically stable cycle

this case the solution corresponding to Γ is asymptotically stable. It seems intuitively acceptable that only ecologically stable cycles can persist in nature. An ecologically stable cycle is necessarily an isolated cycle though it can be seen easily that an isolated cycle need not be ecologically stable. The example

$$\frac{dr}{dt} = r\,(r^2 - 1)\,, \qquad \frac{d\Theta}{dt} = 1 \tag{35}$$

illustrates this fact. The solution is